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UNIFORM ZARISKI'S THEOREM ON FUNDAMENTAL GROUPS

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ABSTRACT

The Zariski theorem says that for every hypersurface in a complex projective (resp. affine) space and for every generic plane in the projective (resp. affine) space the natural embedding generates an isomorphism of the fundamental groups of the complements to the hypersurface in the plane and in the space. If a family of hypersurfaces depends algebraically on parameters then it is not true in general that there exists a plane such that the natural embedding generates an isomorphism of the fundamental groups of the complements to each hypersurface from this family in the plane and in the space. But we show that in the affine case such a plane exists after a polynomial coordinate substitution.

I. Introduction

In [3] Zariski proved the following remarkable theorem. Let $\tilde{H}$ be an algebraic hypersurface in $\mathbb{CP}^n$ where $n \geq 3$. Then for a generic projective plane $\tilde{A} \hookrightarrow \mathbb{CP}^n$ the embedding $\tilde{A} - \tilde{H} \hookrightarrow \mathbb{CP}^n - \tilde{H}$ generates an isomorphism $\pi_1(\tilde{A} - \tilde{H}) \to \pi_1(\mathbb{CP}^n - \tilde{H})$ of the fundamental groups. This implies the similar fact for an algebraic hypersurface H in the Euclidean space $\mathbb{C}^n$. Consider a family of hypersurfaces $\tilde{H}_p \subset \mathbb{CP}^n$ (resp. $H_p \subset \mathbb{C}^n$) depending algebraically on parameter p from an algebraic variety P. It is natural to ask whether there exists a projective plane $\tilde{A}$ (resp. an affine plane A) such that the embedding $\tilde{A} - \tilde{H}_p \hookrightarrow \mathbb{CP}^n - \tilde{H}_p$ (resp.

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$A - H_p \hookrightarrow \mathbb{C}^n - H_p$) generates an isomorphism of the corresponding fundamental groups for every p. In both cases the answer is negative, since it is enough to consider the set of all hyperplanes as this family of hypersurfaces depending algebraically on parameter. But the affine case has some advantage. Namely, we can change the coordinate system in $\mathbb{C}^n$ using a polynomial coordinate substitution (i.e., we can change the set of planes), whenever in the projective case we can use linear substitutions only. This observation leads to the main result of this paper.

For every family of algebraic hypersurfaces H_p in $\mathbb{C}^n$ depending algebraically on parameter $p \in P$, one can choose a coordinate system in $\mathbb{C}^n$ in such a way that for some plane $A_0 \subset \mathbb{C}^n$ the embedding $A_0 - H_p \hookrightarrow \mathbb{C}^n - H_p$ generates an isomorphism $\pi_1(A - H_p) \to \pi_1(\mathbb{C}^n - H_p)$ for each p.

The scheme of the proof can be described as follows. The matter can be reduced to the three-dimensional case. Let $\rho : \mathbb{C}^3 \to \mathbb{C}^2$ be a projection to an (x, y)-plane, let A_0 be $\rho^{-1}(L^0)$ where L^0 is the line $\{y = 0\}$ in the (x, y)-plane. In section 3 we establish when the embedding $A_0 - H \subset \mathbb{C}^3 - H$ generates an isomorphism of the fundamental groups. It turns out that the following conditions are sufficient.
(1) The embedding $\rho^{-1}(o) - H \subset A_0 - H$ generates an epimorphism of the fundamental groups where o is the origin of the (x, y)-plane.
(2) The mapping $\rho\,|_H : H \to \mathbb{C}^2$ is finite and, if it is l-sheeted, then the mapping $\rho\,|_{H\cap A} : H \cap A \to L^0$ is also l-sheeted.
(3) There exists a nonzero polynomial $g \in \mathbb{C}[x]$ such that $g(0) = 0$ and for generic $c \in \mathbb{C}$ the number of points where the curve $L^c = \{y = cg(x)\}$ meets the image Γ of the ramification set of the mapping $\rho\,|_H$ coincides with the number of points where L^0 meets Γ.

The proof of this fact is a modification of the original argument of Zariski who dealt actually with the case when $g(x) = x$, i.e. $\{L^c\}$ is a pencil of lines.

The trouble with the linear coordinate substitutions is that those of them for which one of these three conditions does not hold form a subvariety of codimension 1 in the space of the linear coordinate substitutions. Therefore, for every family of hypersurfaces we shall construct a wider space of polynomial coordinate systems so that for every hypersurface from our family the subset of the coordinate systems in this space with one of these three conditions violated (relative to this hypersurface) has codimension at least l where the "bigger" space of coordinate systems we choose the larger l is. When l is larger than the dimension of the parameter set P we can find a coordinate system such that the embedding $A_0 - H_p \subset \mathbb{C}^3 - H_p$ generates an isomorphism of the fundamental groups for every p.

The paper is organized as follows. After preliminaries (section 2) we show that the embedding $A_0 - H \hookrightarrow \mathbb{C}^n - H$ generates an isomorphism of the fundamental groups provided (1)–(3) hold. In section 4 we prove some technical fact which enable us to describe coordinate systems for which conditions (1)–(3) are true. In the last section we prove the main result for the three-dimensional case first and then we reduce the general case to the three-dimensional one.

2. Terminology and notation

In this paper P and later Q denote always sets of parameters, and a set of parameters is always an algebraic variety over the field of complex numbers.

Definition 2.1: Suppose that T is a closed algebraic subvariety in $\mathbb{C}^n \times P$. Let $\rho_1 \colon T \to \mathbb{C}^n$ and $\rho_2 \colon T \to P$ be the natural projections and let $H_p = \rho_1 \circ \rho_2^{-1}(p)$ where $p \in P$. Then we say that $\{H_p\}$ is a family of algebraic varieties with parameter $p \in P$. We shall mostly deal with the case when H_p is a closed algebraic hypersurface in $\mathbb{C}^n$ for every $p \in P$. In this case we say that $\{H_p\}$ is a family of hypersurfaces in $\mathbb{C}^n$ with parameter $p \in P$. When $n = 2$ we speak about a family of curves.

Remark 2.2: We do not suppose that P is connected or that P has a pure dimension. This enables us, if necessary, to replace P by a disjoint union of its subvarieties. For example, we can treat P as the disjoint union of the strata of its canonical stratification as a singular algebraic variety. Moreover, consider a function ξ on P with a finite number of values such that the preimage of each of these values is a constructive subset of P. Consider a component P' of P and suppose that ξ is constant on $P' - P'_0$ where P'_0 is a closed proper algebraic subvariety of P'.

Convention 2.3: By the previous remark we suppose that for every family of hypersurfaces $\{H_p\}$ the degree (or the Newton polygon) of the defining polynomial f_p of H_p is constant on every component of P. Each polynomial f_p can be represented as $\prod_{i=1}^{l(p)} (f_{p,i})^{n_i(p)}$ where $f_{p,i}$ are irreducible polynomials which are non-proportional for different values of i, and $n_i(p)$ are natural. Using Remark 2.2, we suppose again that the functions $l(p)$ and $n_i(p)$ are constants on every component of P. Furthermore, since we consider the fundamental group of the complement of a hypersurface it is natural to restrict ourselves to the case when this hypersurface is reduced. Thus we can suppose and we *will suppose* that every hypersurface in our family is reduced (that is, we replace $f_p = \prod_{i=1}^{l(p)} (f_{p,i})^{n_i(p)}$ by

$\prod_{i=1}^{l(p)} f_{p,i}$ on each component of P).

Remark 2.4: We say that an algebraic variety B depends algebraically on a polynomial f if B can be viewed as a member of a family of algebraic varieties whose parameter set is a Zariski open subset of the set of polynomials of fixed degree (or with a fixed Newton polygon).

Definition 2.5: Let H be a reduced algebraic hypersurface in $\mathbb{C}^n$ and A be a closed affine algebraic subvariety in $\mathbb{C}^n$. We say that A is H compatible if the embedding $i\colon A - H \hookrightarrow \mathbb{C}^n - H$ generates an epimorphism of the fundamental groups $i_*\colon \pi_1(A-H) \to \pi_1(\mathbb{C}^n - H)$. We say that A is strictly H compatible if i_* is an isomorphism. We say that A is (strictly) compatible relative to a family of hypersurfaces if it is (strictly) H compatible for every hypersurface H in this family.

Definition 2.6: We say that some property holds for a generic point of an algebraic variety P if for every irreducible component P' of P there exists a proper closed algebraic subvariety P'_0 such that this property is true for every $p \in P' - P'_0$.

In particular if we have several properties and each of them holds for a generic point of P, then all of them hold simultaneously for a generic point of P.

3. Modification of Zariski's approach for the non-linear case

In this section the projection $\rho\colon \mathbb{C}^3 \to \mathbb{C}^2$ is given by $(x,y,z) \to (x,y)$ and $H \subset \mathbb{C}^3$ is a reduced hypersurface whose defining polynomial is f. Denote by A_0 the (x,z)-plane in $\mathbb{C}^3$ and by L^0 the x-axis in the (x,y)-plane (of course, $A_0 = \rho^{-1}(L^0)$). Suppose that $g \in \mathbb{C}[x]$ is a nonzero polynomial such that $g(0) = 0$.

Definition 3.1: We say that the plane A^0 is appropriate with respect to the triple (H, ρ, g) if the following properties hold:
(1) The z-axis (i.e., the line $\rho^{-1}(o)$ where o is the origin of the (x,y)-plane) is $H \cap A_0$ compatible in the plane A_0.
(2) The mapping $\rho\,|_H : H \to \mathbb{C}^2$ is finite and, if it is l-sheeted, then the mapping $\rho\,|_{H\cap A_0} : H \cap A_0 \to L^0$ is also l-sheeted.
(3) For generic $c \in \mathbb{C}$ the number of points where the curve $L^c := \{y = cg(x)\}$ meets the image Γ of the ramification set of the mapping $\rho\,|_H$ coincides with the number of points where Γ meets L^0.

The aim of this section is

THEOREM 3.2: *Let A^0 be appropriate with respect to (H, ρ, g). Then A_0 is strictly H compatible.*

In the case when g is linear this theorem can be extracted from the original paper of Zariski [3]. In general case we follow also the outline of his arguments. The proof consists of several lemmas. We discuss first the idea of Zariski's approach and fix some additional notation for the rest of the section.

Let X be the set of points where L^0 meets $\{g(x) = 0\}$. The family $\{L^c\}$ can be viewed as a linear system of curves whose base point set is X. This linear system of curves generates the mapping $\tau: \mathbb{C}^2 - X \to \mathbb{CP}^1$ such that $\tau^{-1}(c) = L^c - X$ for $c \in \mathbb{CP}^1$ (where $L^\infty = \{g(x) = 0\}$). There exists a finite set $C = \{c_1, ..., c_r\} \subset \mathbb{C}$ such that for every $c \in \mathbb{C} - C$ the curve $\tau^{-1}(c)$ meets Γ at the same number of points. Put $Y = \tau^{-1}(\mathbb{C} - C) \cup X$. For every subset K of $\mathbb{C}^2$ we denote by $\tilde{K}$ the set $\rho^{-1}(K) - H$. For instance, $\tilde{Y} = \rho^{-1}(Y) - H$. By condition (3) in Definition 3.1, C does not contain 0. Since $\tau^{-1}(0) = L^0 - X$ the set $\tilde{Y}$ contains $A_0 - H$. Let $\sigma_1, \ldots, \sigma_r$ be a bouquet of simple loops in $\mathbb{C} - C$ with one common point at the origin so that these loops generate the fundamental group of $\mathbb{C} - C$. Put $B = \tau^{-1}(\bigcup_k \sigma_k) \cup X$.

Following Zariski we shall consider the sequence of the embeddings

$$A_0 - H \hookrightarrow \tilde{B} \hookrightarrow \tilde{Y} \hookrightarrow \mathbb{C}^3 \setminus H.$$

Our aim is to prove that each of them generates an isomorphism of the fundamental groups which yields Theorem 3.2. In brief the argument will be as follows. The fact that the first embedding generates an isomorphism is mostly due to van Kampen's theorem [2]. The second embedding generates an isomorphism since $\tilde{B}$ is a deformation retract of $\tilde{Y}$ which is a consequence of Lemmas 3.5 and 3.6 below. The last embedding generates an isomorphism of the fundamental groups by the following reason. The constructive set $\tilde{Y}$ is obtained from $\mathbb{C}^3 \setminus H$ by deleting a *non-closed* hypersurface. This fact enables us to show that for every two-cell in $\mathbb{C}^3 \setminus H$ whose boundary does not meet the closure of this hypersurface, there exists a two-cell in $\tilde{Y}$ with the same boundary. This implies that the induced mapping of the fundamental groups is a monomorphism. It is obvious that it is also surjective.

Let us proceed now with more details. Put $Z = X \cup \tau^{-1}(C \cup \infty)$ (i.e. $Z = L^{c_1} \cup \cdots \cup L^{c_r} \cup L^\infty$). Set $\Gamma' = \Gamma - Z$ (i.e. $\Gamma' = \Gamma \cap \tau^{-1}(\mathbb{C} - C)$) and $\Gamma'' = \rho^{-1}(\Gamma') \cap H$. The set of points where a generic curve L^c meets Γ non-normally is contained in X, by condition (3). Thus every non-smooth point of Γ is contained in Z since at these points none of the generic $\mathbb{C}$-curves L^c can meet Γ normally whence Γ' is smooth.

We need to show that the mapping $\rho\,|_{\Gamma''}: \Gamma'' \to \Gamma'$ is unramified. This fact can be checked locally.

LEMMA 3.3: *Let $\mathcal{H}$ be the germ of an analytic surface at the origin of $\mathbb{C}^3$. Suppose that the mapping ρ_0: $\mathcal{H} \to (\mathbb{C}^2, o)$ is finite where $\rho_0 = \rho\,|_{\mathcal{H}}$. Let $\gamma \subset (\mathbb{C}^2, o)$ be the image of the ramification set for the mapping ρ_0 and let $\mathcal{K} = \rho_0^{-1}(\gamma)$. Suppose that γ is smooth and $\rho_{\mathcal{K}}$: $\mathcal{K} \to \gamma$ is the restriction of ρ to $\mathcal{K}$. Then $\rho_{\mathcal{K}}$ is unramified.*

Proof: Let $\mathcal{L}$ be the preimage of a generic point $b \in (\mathbb{C}^2, o) - \gamma$ under the mapping ρ_0. The fundamental group of $(\mathbb{C}^2, o) - \gamma$ is isomorphic to the group of integers since γ is smooth and this group acts on $\mathcal{L}$. Hence $\mathcal{L}$ can be represented as the disjoint union of minimal invariant subsets of $\mathcal{L}$ relative to this action. These subsets correspond to the irreducible components of $\mathcal{H}$. Let s be the number of points in the preimage $\mathcal{L}_0$ of a generic point $a \in \gamma$ under ρ_0. Sending b to a one can see that the points of $\mathcal{L}_0$ generate a partition of $\mathcal{L}$ into disjoint subsets. Using the fact that the fundamental group of $(\mathbb{C}^2, o) - \gamma$ is the group of integers, one can check that this partition is invariant under the action of the fundamental group (it is enough to consider the action on $\mathcal{L}$ of a small simple loop from $(\mathbb{C}^2, o) - \gamma$ around a since this loop can be viewed as a generator of the fundamental group). Hence we can represent $\mathcal{H}$ as the union $\bigcup_{i=1}^{s} \mathcal{H}_i$ of the germs of surfaces so that $\mathcal{H}_i \cap \rho^{-1}(\gamma) = \mathcal{K}_i$ where $\mathcal{K}_i \subset \mathcal{K}$ are the germs of different curves with the origin as the only common point (since $\mathcal{K}_i \cap \rho^{-1}(a)$ is exactly one point in $\mathcal{L}_0$ that corresponds to $\mathcal{H}_i$). Assume that $s \geq 2$. Consider the germ of the curve $\zeta = \mathcal{H}_1 \cap \mathcal{H}_2$. By construction, the germ $\rho(\zeta)$ meets γ at the origin only. But it must be contained in γ since ζ is contained in the ramification set. Contradiction. Hence $s = 1$ which is the desired conclusion. ∎

LEMMA 3.4: *The natural embedding i: $\tilde{Y} \hookrightarrow \mathbb{C}^3 - H$ generates an isomorphism of the fundamental groups i_*: $\pi_1(\tilde{Y}) \to \pi_1(\mathbb{C}^3 - H)$.*

Proof: By condition (3) in Definition 3.1, C does not contain 0. Since $\tau^{-1}(0) = L^0 - X$ the set $\tilde{Y}$ contains $A_0 - H$. We shall see later (Lemma 4.2) that under condition (2) every line which is parallel to the z-axis and which meets H at l points, is H compatible. Thus condition (2) implies that A_0 is H compatible. Hence i_* is an epimorphism.

Let $\delta_1, \dots, \delta_s$ be a set of generators in $\pi_1(\tilde{Y})$, and, therefore, it can be treated as a set of generators in $\pi_1(\mathbb{C}^3 - H)$. We need to show that if $\delta_1, \dots, \delta_s$ satisfy some generating relations in $\pi_1(\mathbb{C}^3 - H)$ then they satisfy the same relations in $\pi_1(\tilde{Y})$. For this purpose it suffices to show that for every 2-cell Δ_1 in $\mathbb{C}^3 - H$ with a boundary $\partial\Delta_1 \subset \tilde{Y}$ there exists a 2-cell $\Delta_2 \subset \tilde{Y}$ with the same boundary. We can suppose that if u is an intersection point of $\tilde{Z}$ and the interior of Δ_1 then Δ_1 meets $\tilde{Z}$ normally at u, and $u \notin \tilde{X}$. (We can do this since the real codimension of $\tilde{X}$ in $\mathbb{C}^3 - H$ is 4.) Since X is the base point set for $\{L^c\}$ one can choose a path ξ in $\tilde{Z}$ joining u and a point $v \in \tilde{X}$ in such a manner that $\rho(v)$ is the only point from $\rho(\xi)$ that belongs to X and that $\xi - v$ is contained in the smooth part $\tilde{Z}^*$ of $\tilde{Z}$. One can identify a neighborhood of $\tilde{Z}^*$ in $\mathbb{C}^3$ with a neighborhood of the zero section of the normal bundle to $\tilde{Z}^*$. Moreover, we can suppose that the intersection of Δ_1 with this neighborhood is contained in a fiber of this bundle. Choose a small 2-cell $\Delta_\varepsilon(u) \subset \Delta_1$ with center at u and replace it by a cone in $\tilde{Y}$ with the following properties: v is the vertex of the cone and the only point where the cone meets $\tilde{Z}$, the base of the cone coincides with the boundary of $\Delta_\varepsilon(u)$, the intersection of the cone with the fiber of the normal bundle to $\tilde{Z}^*$ at every point of $\xi - v$ is a circle. Repeating this procedure we obtain a 2-cell $\Delta_2 \subset \mathbb{C}^3 - H$ such that its boundary coincides with the boundary of Δ_1 and its interior meets $\tilde{Z}$ only at points from $\tilde{X}$. In particular, $\Delta_2 \subset \tilde{Y}$. ∎

Recall that $\sigma_1, \dots, \sigma_r$ is a bouquet of simple loops in $\mathbb{C} - C$ with one common point at the origin so that these loops generate the fundamental group of $\mathbb{C} - C$, and $B = \tau^{-1}(\bigcup_k \sigma_k) \cup X$. We have to show that $\tilde{B}$ is a deformation retract of $\tilde{Y}$, and, in particular, the embedding $j\colon \tilde{B} \hookrightarrow \tilde{Y}$ generates an isomorphism $j_*\colon \pi_1(\tilde{B}) \to \pi_1(\tilde{Y})$. The construction of this deformation can be reduced to a simpler problem due to

Lemma 3.5: *Let K be a subset of Y. Suppose that $\kappa = \{\kappa_t \mid t \in [0,1]\}$ is a path in the space of continuous mappings from K to Y such that κ_0 is the identical embedding, the restriction of κ_t to $K \cap X$ is the identical embedding for every $t \in [0,1]$, $\kappa_t(\Gamma \cap K) \subset \Gamma$, and $\kappa_t(K - (X \cup \Gamma)) \subset Y - (X \cup \Gamma)$. Suppose also that the restriction of κ to $(K - X) \times [0,1]$ is smooth. Then there exists a deformation $D = \{D_t \mid t \in [0,1]\}$ of the identical embedding $D_0 : \tilde{K} \hookrightarrow \tilde{Y}$ such that $\rho D = \kappa\rho$ and D is identical on $\tilde{K} \cap \tilde{X}$. Moreover, if for every t the mapping κ_t is a homeomorphism between K and $\kappa_t(K)$ then D_t is a homeomorphism between $\tilde{K}$ and $D_t(\tilde{K})$.*

Proof: Fix a neighborhood U of $H - \rho^{-1}(X)$ in $\mathbb{C}^3$ so that $\bar{U} \cap \rho^{-1}(b)$ is compact for every $b \in \mathbb{C}^2$ ($\bar{U}$ is, of course, the closure of U in $\mathbb{C}^3$). Suppose also that $\bar{U} \cap \rho^{-1}(X) = H \cap \rho^{-1}(X)$, i.e. this set is finite. Consider the natural projection

$T\mathbb{C}^2 \to \mathbb{C}^2$ where $T\mathbb{C}^2$ is the tangent bundle of $\mathbb{C}^2$, and the mapping $\rho\colon \mathbb{C}^3 \to \mathbb{C}^2$. They generate the set $T = \mathbb{C}^3 \otimes_{\mathbb{C}^2} T\mathbb{C}^2$ with the natural projections $pr_1\colon T \to \mathbb{C}^3$ and $pr_2\colon T \to T\mathbb{C}^2$. Put $Y' = \mathbb{C}^3 - (H \cap \rho^{-1}(X \cup \Gamma))$ and put $W = pr_1^{-1}(Y')$. Let $W_0 = pr_1^{-1}(\tilde{\Gamma}') \cap pr_2^{-1}(T\Gamma')$ and $\overline{W}_0 = pr_1^{-1}(\rho^{-1}(\Gamma')) \cap pr_2^{-1}(T\Gamma')$ where $T\Gamma'$ is the tangent bundle to Γ'. Using partition of unity one can construct a smooth mapping $\chi\colon W \to TY'$ where TY' is the tangent bundle of Y' with the following properties:

(i) for every $w \in W$ we have $\rho_* \chi = pr_2$;

(ii) for every $w \in W$ with $pr_1(w) \notin U$ the z-coordinate of the vector $\chi(w)$ is zero;

(iii) for every $w \in W$ with $pr_1(w) \in H$ the vector $\chi(w)$ is tangent to H (note that this tangent vector exists since the restriction of ρ to $H - \rho^{-1}(\Gamma)$ is an unramified covering of $\mathbb{C}^2 - \Gamma$);

(iv) the restriction of χ to W_0 can be extended to a smooth mapping $\chi_0\colon \overline{W}_0 \to T\mathbb{C}^3$ (where $T\mathbb{C}^3$ is the tangent bundle of $\mathbb{C}^3$) so that for every $w \in \overline{W}_0$ with $pr_1(w) \in \Gamma''$ the vector $\chi_0(w)$ is tangent to Γ''. (This vector $\chi_0(w)$ exists since the restriction of ρ to Γ'' is an unramified covering of Γ', by Lemma 3.3.)

It is worth mentioning that we need condition (iv) separately from (iii) since in general the mapping χ cannot be extended continuously to the points w with $pr_1(w) \in \rho^{-1}(\Gamma) \cap H$.

Consider the curve $\kappa(b) = \{\kappa_t(b) \mid t \in [0,1]\}$ for $b \in K - X$. Suppose for simplicity that it has no selfintersection points (otherwise we can replace the curve with its graph). Then $\widetilde{\kappa(b)}$ is a smooth real manifold. For each $u_0 \in \widetilde{\kappa(b)}$ there exists $t_0 \in [0,1]$ such that $\rho(u_0) = \kappa_{t_0}(b)$. Let $v_b(t_0)$ be the vector tangent to $\kappa(b)$ at $\kappa_{t_0}(b)$ which is generated by differentiation with respect to t. Then the vector $\chi(u_0 \otimes v_b(t_0))$ is tangent to $\widetilde{\kappa(b)}$ at u_0. Therefore, such vectors define a vector field on $\widetilde{\kappa(b)}$. Let $\rho(u) = b$ and let $D(u) = \{D_t(u) \mid t \in [0,1]\}$ be the integral curve of this vector field such that it begins at the point $u = D_0(u)$. These curves define a deformation D of $\tilde{K} - \tilde{X}$ in $\tilde{Y} - \tilde{X}$ with $\rho D = \kappa\rho$ unless for some $u \in \tilde{K} - \tilde{X}$ the curve $D(u)$ goes either to infinity or to H for a finite time. It cannot go to infinity for a finite time within U due to the fact that $\bar{U} \cap \rho^{-1}(b)$ is compact for every b and the mapping $\rho\mid_H$ is finite. Outside U it cannot go to infinity as well, by (ii). When $b \notin \Gamma$ the curve $D(u)$ cannot reach H for a finite time due to (iii). When $b \in \Gamma$ the curve $\kappa(b) \subset \Gamma$, by the assumption of Lemma, and $D(u)$ cannot reach H again for a finite time due to (iv). Therefore, $D(u) \subset \tilde{Y}$ and $D_t(\tilde{K} - \tilde{X}) \subset \tilde{Y}$ for every t.

Note that the combination of (ii) and the facts that $\kappa_t\mid_{K\cap X}$ is the identical embedding and that the set $\bar{U} \cap \rho^{-1}(X)$ is finite implies that D can be extended to $\tilde{X}$ by the identical deformation which proves the first statement of Lemma. The second statement follows obviously from the construction of D. ∎

The next lemma is almost the exact repetition of Lemma 3.5 but we give its proof for the sake of completeness.

LEMMA 3.6: *Let $K \subset Y$ and $\kappa_0: K \to Y$ be the identical embedding. Let θ_0 be the identical embedding of $\tau(K - X)$ into $\mathbb{C} - C$. Suppose that there exists a smooth path $\theta = \{\theta_t | t \in [0, 1]\}$ in the space of smooth mappings from $\tau(K - X)$ to $\mathbb{C} - C$. Then there exist a deformation $\kappa = \{\kappa_t\}$ of κ_0 so that $\kappa_t |_X$ is the identical embedding for every t, $\kappa(\Gamma \cap K) \subset \Gamma$, $\kappa(K - (X \cup \Gamma)) \subset Y - (X \cup \Gamma)$, and $\tau \circ \kappa|_{K-X} = \theta \circ \tau|_{K-X}$. Moreover, if for every t the mapping θ_t is a diffeomorphism between $\tau(K - X)$ and its image then κ_t is a homeomorphism between K and its image, and the restriction of κ to $(K - X) \times [0, 1]$ is smooth.*

Proof: Note that $Y - X = \tau^{-1}(\mathbb{C} - C)$ can be treated as $(\mathbb{C} - \{x_1^0, \ldots, x_l^0\}) \times (\mathbb{C} - C) \subset \mathbb{C} \times (\mathbb{C} - C)$ where $x_1^0, \ldots, x_l^0$ are the x-coordinates of the points of X. Denote the image of Γ' in $\mathbb{C} \times (\mathbb{C} - C)$ by the same symbol Γ'. Note that Γ' is closed in $\mathbb{C} \times (\mathbb{C} - C)$ and does not meet any of sets $x_i^0 \times (\mathbb{C} - C)$. Indeed, otherwise it contains a point (x_i^0, c^0) where $c^0 \notin C$. This implies that the curve L^{c^0} (and, therefore, every curve L^c) meets Γ at the corresponding point of X. Furthermore, this implies that the local intersection number of L^{c^0} and Γ at this point is greater than the similar number for L^c and Γ where c is generic. Hence L^{c^0} meets Γ at a fewer number of points than a generic L^c does which contradicts the description of C. Note that τ can be treated as the natural projection τ_2 to the second factor. Let $\tau_1 : \tau^{-1}(\mathbb{C} - C) \to \mathbb{C}$ be the projection to the first factor. Choose a small tubular neighborhood U of Γ' in $\tau^{-1}(\mathbb{C} - C)$ such that its closure $\bar{U}$ in $\mathbb{C} \times (\mathbb{C} - C)$ does not meet the sets $x_i^0 \times (\mathbb{C} - C)$, $i = 1, \ldots, l$. Using partition of unity we can construct a vector field μ on $\mathbb{C} \times (\mathbb{C} - C)$ so that

— outside U we have $\tau_{1*}(\mu) \equiv 0$,

— μ is tangent to Γ',

— $\tau_{2*}(\mu)$ is a nonzero constant vector field on $\mathbb{C} - C$ (this means that one can suppose that the phase flow associated with μ transforms L^c into L^{c+t} for time t whenever this flow is defined correctly).

If $b \in K \cap X$ put $\kappa(b) = b$ and if $b \in K - X$ define $\kappa_t(b)$ as follows. Consider $c = \tau(b)$ and $M_c = \tau^{-1}(\theta(c))$. Suppose for simplicity that $\theta(c)$ has no selfintersection points. Then M_c is a smooth real manifold which is naturally embedded in $\tau^{-1}(\mathbb{C} - C)$. For each $a \in M_c$ there exists t such that $\tau(a) = \theta_t(c)$. Consider the vector $\theta_t'(c)\mu(a)$ at a where $\theta_t'(c)$ is the derivative of the function $\theta(c)$: $[0,1] \to \mathbb{C}$ with respect to t. This vector is tangent to M_c and, therefore, such vectors define a vector field μ_c on M_c. This vector field defines uniquely an integral curve $\kappa_t(b)$ in M_c which begins at $b = \kappa_0(b)$.

The continuity of κ is clear unless for some $b \in K - X$ the curve $\kappa(b)$ goes either to infinity or to $x_i^0 \times (\mathbb{C} - C)$ for a finite time. But it cannot go to infinity since we cannot reach infinity within U for a finite time due to the description of U, and outside U the behavior of $\kappa_t(b)$ is defined by the vector field μ which does not send points from $\tau^{-1}(\mathbb{C} - C)$ to infinity since $\tau_{1*}(\mu) = 0$. Similarly, a set $x_i^0 \times (\mathbb{C} - C))$ cannot be reached for a finite time. Thus κ is continuous and $\kappa_t(K - X) \subset Y - X$ for every t. Note that μ_c is tangent to $\Gamma \cap M_c$. Hence the curve $\kappa(b)$ is either contained in Γ or does not meet it which yields the desired properties of κ in the first statement. The second statement follows obviously from the construction of κ and the fact that the integral curve $\kappa(b)$ depends smoothly on $b \in K - X$. ∎

Proof of Theorem 3.2: We can suppose that the bouquet $\bigcup_k \sigma_k$ is chosen so that there exists a smooth deformation θ of $\mathbb{C} - C$ to this bouquet. By Lemma 3.6, there exists a deformation $\kappa = \{\kappa_t\}$ of Y to B with the prescribed properties. By Lemma 3.5, there exists a deformation of $\tilde{Y}$ to $\tilde{B}$ which is identical on $\tilde{X}$. Thus it remains to show that the natural embedding of $A_0 - H$ into $\tilde{B}$ generates an isomorphism of the fundamental groups. As we mentioned in the proof of Lemma 3.4 this embedding generates an epimorphism of the fundamental groups. By (1) the line $\rho^{-1}(o)$ is H compatible. Hence the loops $\delta_1, \ldots, \delta_s$ which generate $\pi_1(\rho^{-1}(o) - H)$ can be viewed as generators of $\pi_1(\tilde{B})$. If we travel along a loop σ_j: $[0,1] \to \mathbb{C} - C$ from the point $o = \sigma_j(0)$ to a point $c = \sigma_j(t)$ then Lemma 3.6 provides us with an appropriate homeomorphism between L^0 and L^c which depends continuously on t. It generates in turn a homeomorphism (depending continuously on t) between $\tilde{L}^0$ $(= A_0 - H)$ and $\tilde{L}^c$ which is identical on $\tilde{X} = \tilde{L}^0 \cap \tilde{L}^c$, by Lemma 3.5. Thus after traveling along the whole loop σ_j we deform each element $\delta \in \pi_1(\tilde{L}^0)$ into another element $\delta^{(j)} \in \pi_1(\tilde{L}^0)$. By the van Kampen theorem [2], the generating relations for $\delta_1, \ldots, \delta_s$ in $\pi_1(\tilde{L}^0)$ together with the relations $\delta_i = \delta_i^{(j)}$ give all the generating relations between $\delta_1, \ldots, \delta_s$ in $\pi_1(\tilde{B})$. But since for every j the homeomorphism of $\tilde{L}^0$ generated by the loop σ_j is identical on $\tilde{X}$ the loops δ_i and $\delta_i^{(j)}$ coincide for every i. Hence the embedding of $\rho^{-1}(o) - H$ into $\tilde{B}$ generates an isomorphism of the fundamental groups. Since $\tilde{B}$ is a deformation retract of $\tilde{Y}$, Lemma 3.4 implies that the embedding of $A_0 - H$ into $\mathbb{C}^3 - H$ generates also an isomorphism of the fundamental groups. ∎

4. Technical facts

In this section we shall study polynomial coordinate substitutions which provide conditions (1)–(3) from Definition 3.1. Condition (3) requires most effort and in order to find an appropriate coordinate substitution we face the following problem.

Let Γ be a closed affine curve in the (x,y)-plane and L_g be a curve given by $y+h^0(x)+g(x)=0$ where h^0 is a fixed polynomials and g is from the set G_m of polynomials of degree at most m. Describe reasonable assumptions on g under which the curve L_g is Γ compatible. Similarly, if $\{\Gamma_p|p\in P\}$ is an algebraic family of plane affine curves, then when is L_g compatible with respect to each member of this family?

We present first the outline of our arguments. For every closed affine curve $\Gamma\subset\mathbb{C}^2$ one can suppose that after a coordinate substitution the projection of Γ to the x-axis is finite (see Lemma 4.1). After this substitution every line $x=c$, $c\in\mathbb{C}$ which meets Γ at the maximal number of points (counting without multiplicity) is Γ compatible (Lemma 4.2). Then we note that if an algebraic family of curves contains an element L which has a smooth reduced Γ compatible component then a generic member of this family is Γ compatible. In our particular case this component of L is the line $x=c$ (Lemma 4.3) and the other curves from the family are of form L_g where g runs over a subvariety in G_m. Thus it should be understood when a curve L_{g^0} can be included in such a family as a generic element. When the degree of h^0 is large enough the intersection number $\Gamma\cdot L_g$ does not depend on g (Lemma 4.6). The difference between this intersection number and the number of points where L_g meets Γ normally is called the defect of Γ relative to L_g. It is shown in Proposition 4.8 that if the defect of Γ relative to L_{g^0} is at most $m-2$ then one can construct a family of curves as above in which L_{g^0} is a generic element (that is, L_{g^0} is Γ compatible). Furthermore, consider those g's for which the defect of Γ relative to L_g is at least $m-2$. Proposition 4.8 shows that they form a subvariety of G_m whose codimension increases when m increases. This enables us to find a curve L_{g^0} which is Γ_p compatible for every $p\in P$, and, therefore, to answer the problem above.

The next two lemmas are formulated for hypersurfaces in $\mathbb{C}^3$ but they can be easily reformulated for hypersurfaces in $\mathbb{C}^n$ with $n\geq 3$.

LEMMA 4.1: *Let (x_1,y_1,z_1) be a coordinate system in $\mathbb{C}^3$ and let*

$$\{H_p\subset\mathbb{C}^3\mid p\in P\}$$

be a family of hypersurfaces with defining polynomials $\{f_p\}$. Suppose that d is *natural such that*

$$d > \max_{p \in P} \deg f_p.$$

Let (x,y,z) be a new coordinate system such that

$$(x_1, y_1, z_1) = (g_1(x,y,z), g_2(x,y,z), z)$$

where g_1 and g_2 are polynomials. Suppose that for every constant a and b the degrees of $g_1(a,b,z)$ and $g_2(a,b,z)$ are d_1 and d_2 respectively. Let $d_2 > d$ and $d_1 > d \cdot d_2$. Then the restriction of the projection ρ (given by $(x,y,z) \to (x,y)$) to every hypersurface H_p is a finite morphism.

Proof: Fix $p \in P$. The restriction of ρ to H_p is finite if for every constant a and b the degree of the polynomial

$$\varphi(z) = f_p(g_1(a,b,z), g_2(a,b,z), z)$$

does not depend on a and b. Consider monomials $x_1^k y_1^l z_1^m$ which are present in f_p with nonzero coefficients and consider the vectors (k,l,m). Suppose that (k^0, l^0, m^0) is the greatest among these vectors in the lexicographic order. One can see that $\deg \varphi$ coincides then with $k^0 d_1 + l^0 d_2 + m$ regardless of the choice of a and b. ∎

LEMMA 4.2: *Let ρ: $\mathbb{C}^3 \to \mathbb{C}^2$ be the projection given by $(x,y,z) \to (x,y)$. Suppose that a reduced hypersurface H in $\mathbb{C}^3$ does not contain lines parallel to the z-axis. Consider a line $L = \rho^{-1}(w)$ with $w \in \mathbb{C}^2$ which meets H at $\deg_z f$ points (counting without multiplicity). Then L is H compatible.*

Proof: There is an algebraic subvariety $S \subset \mathbb{C}^2$ such that the line $\rho^{-1}(s)$ meets H at less than $\deg_z f$ points counting without multiplicity iff $s \in S$. Put $E = \mathbb{C}^3 - (H \cup \rho^{-1}(S))$ and $\tau = \rho\,|_E$. Then the mapping τ: $E \to \mathbb{C}^2 - S$ is a fibration whose generic fiber F is a $\deg_z f$ times punctured complex line. Let i: $F \to E$ be the natural embedding. We have the exact sequence of the fundamental groups

$$\to \pi_1(F) \overset{i_*}{\to} \pi_1(E) \overset{\tau_*}{\to} \pi_1(\mathbb{C}^2 - S) \to 0.$$

Choose simple loops $\{\sigma_i\}$ in $\mathbb{C}^2 - S$ around each point in a finite set $\{s_i\} \subset S$ such that these loops generate the whole fundamental group $\pi_1(\mathbb{C}^2 - S)$. Since $\rho^{-1}(s_i)$ is not contained in H one can choose a loop γ_i in E so that it is contractible in $\mathbb{C}^3 - H$ and $\tau(\gamma_i) = \sigma_i$. Then the exact sequence implies that every element in $\pi_1(E)$ can be written in the form uv where $v \in i_*(\pi_1(F))$ and u lies in the group generated by $\{[\gamma_i]\}$. Consider the embedding $j\colon E \to \mathbb{C}^3 - H$ and the corresponding homomorphism of the fundamental groups $j_*\colon \pi_1(E) \to \pi_1(\mathbb{C}^2 - H)$. Note that $[\gamma_i] \in \ker j_*$ and $j_*(uv) = j_*(v)$. On the other hand j_* is surjective, of course. Hence $j_* \circ i_*(\pi_1(F)) = \pi_1(\mathbb{C}^3 - H)$ and we are done, since $L - H$ is a generic fiber of τ. ∎

We shall cite notation which will be used in the remainder of this section. A curve Γ in $\mathbb{C}^2$ is always reduced and it coincides with the zero locus of a polynomial f. We shall denote by h^0 a polynomial in one variable of degree d_0. We shall consider $\mathbb{C}$-curves in $\mathbb{C}^2$ given by equations of form $y + h^0(x) + g(x) = 0$ where g runs over the set of polynomials G_m of degree at most $m < d_0$. The family of these curves will be denoted by $V_m(h^0)$. There is a natural bijection between G_m and $V_m(h^0)$ and we denote by L_g the $\mathbb{C}$-curve from $V_m(h^0)$ that corresponds to $g \in G_m$.

LEMMA 4.3: *Let Γ be an algebraic curve in $\mathbb{C}^2$ which does not contain lines parallel to the y-axis. Let a polynomial g have a simple root x^0 so that the line $C = \{x = x^0\}$ meets Γ at $\deg_y f$ different points. Let $g^0 \in G_m$ and $g(c) = g^0 + cg$. Then the curve $L_{g(c)} \in V_m(h^0)$ is Γ compatible when $|c|$ is sufficiently large.*

Proof: By Lemma 4.2, C is Γ compatible. When $|c| \to \infty$ then $L_{g(c)}$ approaches C. Repeating the argument of [1, Lemma 3], one can see that $L_{g(c)}$ is Γ compatible for large $|c|$. (In [1, Lemma 3] we used the term "Γ proper" instead of "Γ compatible". We made this replacement since the term "proper" may be misunderstood.) ∎

Remark 4.4: Suppose that $\omega \in G_m$ is a nonzero polynomial and $g^0 \in G_m$. Let V be an affine subspace of $V_m(h^0)$ so that it consists of the $\mathbb{C}$-curves of form $\{y + h^0(x) + g^0(x) + \omega(x)\tilde{g}(x) = 0\}$ where $\tilde{g} \in G_n$ and $n = m - \deg \omega$. Note that if $n > 0$ then for generic $\tilde{g} \in G_n$ the polynomial $g = \tilde{g}\omega$ has always a simple root x^0 such that the line C described in Lemma 4.3 is Γ compatible.

LEMMA 4.5: *Suppose that the curve L_{g^0} meets Γ at the same number of points as a generic $\mathbb{C}$-curve in V where L_{g^0} and V are from Remark 4.4 with $n > 0$. Suppose also that Γ does not contain lines parallel to the y-axis. Then L_{g^0} is Γ compatible.*

Proof: Put $\mathcal{L} = \{((x,y), L) \in \mathbb{C}^2 \times V \mid (x,y) \in L - \Gamma\}$. Let $\kappa\colon \mathcal{L} \to V$ be the natural projection. Then there exists a closed algebraic subvariety S of V such that a curve L from V meets Γ at a fewer number of points than the generic curve from V iff this curve L is from S. Hence the restriction of κ to $\kappa^{-1}(V-S)$ is a fibration over $V-S$. This fibration provides an isotopy between $L_{g^0}-\Gamma$ and $L-\Gamma$ in $\mathbb{C}^2-\Gamma$ where L is a generic curve in V. By Lemma 4.3 and Remark 4.4, L and, therefore, L_{g^0} are Γ compatible. ∎

The number of points at which L_g meets Γ may change when L_g runs over $V_m(h^0)$ but at least we can fix the number of intersection points of L_g and Γ counting with multiplicity.

LEMMA 4.6: *Let $\{\Gamma_p\}$ be a family of curves in $\mathbb{C}^2$ with parameter $p \in P$ and let d_0 be natural so that $d_0 > m+k$ where $k = \dim P$ and $m > 0$. Then for generic $h^0 \in G_{d_0}$ and every $p \in P$ the intersection number $L_g \cdot \Gamma_p$ is finite and it does not depend on $g \in G_m$. In particular, L_g is not a component of Γ_p for every p and g.*

Proof: Let N be the maximal possible degree of the polynomial $\varphi(x) = f(x,h(x))$ where h runs over G_{d_0}. Let L be the curve $y = h(x)$. Suppose that $L\cdot\Gamma = N-l$. Then the degree of $f(x,h(x)) = \sum_{i=0}^{N} a_i x^i$ is $N-l$. Note that the coefficients $a_i = 0$ for $i > N-l$ only when h belongs to an algebraic subvariety $\mathcal{A}(l)$ of G_{d_0} which depends algebraically on f. If the leading coefficient of φ does not depend on h this subvariety is empty. Otherwise its codimension in G_{d_0} is l when $l < d_0$. Replace $\Gamma, f, \mathcal{A}(l)$ by $\Gamma_p, f_p, \mathcal{A}_p(l)$ and consider $\mathcal{B}(l) = \bigcup_{p\in P}\mathcal{A}_p(l)$. Then the closure of $\mathcal{B}(l)$ is an algebraic subvariety in G_{d_0} whose codimension is at least $l-k$. In particular, a generic polynomial h^0 from G_{d_0} does not belong to $\mathcal{B}(k+1)$. Let $h,\tilde{h} \in G_{d_0}$ be such that h has the same $k+1$ leading coefficients as $\tilde{h}$ does. Note that for every $p \in P$ and $l \le k$ we have $h \in \mathcal{A}_p(l)$ iff $\tilde{h} \in \mathcal{A}_p(l)$. Hence for every $p \in P$ and $l \le k$ if $h^0 \in \mathcal{A}_p(l) - \mathcal{B}(k+1)$ then $h^0 + g \in \mathcal{A}_p(l) - \mathcal{B}(k+1)$ for every $g \in G_m$, which is the desired conclusion. ∎

Thus one can suppose further that the function $L_g \cdot \Gamma$ is constant on G_m, i.e. the points from $L_g \cap \Gamma$ do not go to infinity when g runs over G_m. Note also that if g is a small perturbation of g^0 in G_m and L_{g^0} meets Γ normally at some point then L_g meets Γ normally at a nearby point. Therefore, in order to construct a subspace V as in Lemma 4.5 we should take care of the set $\overline{w} = \{w_1, \ldots, w_l\}$ at which Γ meets L_{g^0} non-normally. Let k_i be the local intersection number of Γ and L_{g^0} at w_i and let $k = k_1 + \cdots + k_l$.

Definition 4.7: We call k the defect of L_{g^0} relative to Γ.

Let μ_w be the multiplicity of Γ at a point $w \in \Gamma$. Put $\mu(\Gamma) = \max(\mu_w \mid w \in \Gamma)$ and $\alpha(\Gamma) := \max(4 \log \mu(\Gamma), 1)$.

PROPOSITION 4.8: *Let $\alpha := \alpha(\Gamma)$, and let $S_k(\Gamma)$ be the subset of G_m such that for each $g \in S_k(\Gamma)$ the defect of L_g relative to Γ is at least k. Suppose that*

(1) *$k < m$ and*

(2) *$L_g \cdot \Gamma$ is constant on G_m.*

Then the following are true.

(i) *$S_k(\Gamma)$ is an algebraic subvariety of G_m which depends algebraically on f.*

(ii) *The codimension of $S_k(\Gamma)$ in G_m is more than $(\log k)/\alpha$.*

(iii) *For every $g^0 \notin S_k(\Gamma)$ there exists a nonzero polynomial ω in one variable such that $\deg \omega = k+1$, $\omega(0) = 0$, and every generic $\mathbb{C}$-curve L_g from V described in Remark 4.4 meets Γ at the same number of points as L_{g^0}.*

(iv) *If $k < m - 1$ and Γ does not contain lines parallel to the y-axis then L_g is Γ compatible for every $g \notin S_k(\Gamma)$.*

Proof: Consider $L^h = \{y + h(x) = 0\}$ for every $h \in G_{d_0}$. Let $w^0 \in \Gamma$ and let $1 < k^0 < d_0$. We denote by $T_0(w^0, k^0)$ the subset of G_{d_0} such that for every h from this subset the local intersection number of L^h and Γ at w^0 is at least k^0. Our first aim is to estimate the codimension of $T_0(w^0, k^0)$ in G_{d_0}. Let x^0 be the x-coordinate of w^0 and let $\varphi(x, h) = f(x, -h(x))$. We denote by φ_s the s-th derivative of φ with respect to x. Note that $h \in T_0(w^0, k^0)$ iff

(4.1.0) $$(x^0, h(x^0)) = w^0,$$

(4.1.s) $$\varphi_s(x^0, h) = 0,$$

where $s = 1, \ldots, k^0 - 1$. Let μ_0 be the multiplicity of Γ at w^0. Then the equations (4.1.s) holds automatically for $1 \leq s \leq \mu_0 - 1$. Let $h(x) = \sum_i c_i x^i$. One can rewrite equations (4.1) in the form of polynomial equations on the coefficients $\{c_i\}$,

(4.2.0) $$\Phi_0(c_0) = 0,$$

(4.2.s) $$\Phi_s(\{c_i\}) = 0,$$

where $s = \mu_0, \ldots, k^0 - 1$. Without loss of generality we can suppose that $x^0 = 0$. Then (4.2.0) means that $-c_0$ coincides with the y-coordinate of w^0, and Φ_s is just the coefficient before the monomial x^s in the polynomial $f(x, -h(x))$. Replacing

y by $y + c_0$ we can suppose that w^0 is the origin. Then $c_0 = 0$ and one can see that the codimension of $T_0(w^0, k^0)$ in G_{d_0} is one more than the codimension of the affine algebraic variety given by the equations (4.2.s) which do not contain now the variable c_0. Consider three cases.

CASE 1: $k^0 = \mu_0 \geq 2$. The codimension of $T_0(w^0, k^0)$ in G_{d_0} is $1 > (\log k^0)/\alpha$.

CASE 2: $k^0 > \mu_0 \geq 2$. The polynomial f does not contain monomials of degree less than μ_0. Furthermore, $(x)^{\mu_0}$ cannot be the only monomial of degree μ_0 in f with a nonzero coefficient since $k^0 > \mu_0$. Hence f contains monomials of type $y^r x^{\mu_0 - r}$ (where $r > 0$) with nonzero coefficients. Suppose also that r_0 is the maximum among such r's. Consider in (4.2) the equations with $s = r_0 t + \mu_0 - r_0$ for some natural t. The assumption on the monomials of degree μ_0 in f implies that $\Phi_s = \lambda(c_t)^{r_0} + \Psi_s$ where λ is a nonzero constant and the degree of the polynomial Ψ_s in variable c_t is less than r_0. Denote by E_n the equation (4.2.s) with $s = r_0(\mu_0)^{n-1} + \mu_0 - r_0$ where n can be any number from 1 to $[\log(k^0 - 1)/\log \mu_0]$ ($[a]$ is the entire part of a). We saw already that E_n depends on c_t with $t = (\mu_0)^{n-1}$. On the other hand, E_n does not depend on c_j where $j = (\mu_0)^i$ and $i \geq n$. Hence the codimension of $T_0(w^0, k^0)$ in G_{d_0} is $[(\log(k^0 - 1))/\log \mu_0] > (\log(k^0 - 1))/(2 \log \mu_0) > (\log k^0)/(4 \log \mu_0) \geq (\log k^0)/\alpha$.

CASE 3: $\mu_0 = 1$. One can check that for every s the equation (4.2.s) does not contain c_i when $i > s$ and it contains only a linear term with c_s. Hence the codimension of $T_0(w^0, k^0)$ in G_{d_0} is k^0. That is, it is at least $1 + (\log k^0)/\alpha$ since $k^0 \geq 2$.

Let G be the subset of G_{d_0} that consists of all h of form $h = h^0 + g$ where h^0 is fixed and g runs over G_m. Let $T^0(w^0, k^0) = T_0(w^0, k^0) \cap G$. We need to find the codimension of $T^0(w^0, k^0)$ in G in the case when $k^0 < m$. Suppose that $g(x) = \sum_i b_i x^i$. Note that there is one-to one correspondence between coefficients $\{c_i\}$ and $\{b_i\}$ where $i = 0, \ldots, m$. Since Φ_s does not contain unknowns c_i with $i > k^0$ we see that the codimension of $T^0(w^0, k^0)$ in G is at least $(\log k^0)/\alpha$ for singular w^0 and it is at least $1 + (\log k^0)/\alpha$ when w^0 is regular.

Let $\bar{w} = (w_1, \ldots, w_l)$ be different points on Γ and let $\bar{k} = (k_1, \ldots, k_l)$ where each $k_i \geq 2$. Consider $R_0(\bar{w}, \bar{k}) = \bigcap_{i=1}^l T^0(w_i, k_i)$. Let $\hat{w} = (w_1, \ldots, w_n)$ where $n \leq l$. Suppose that all coordinates of $\hat{w}$ are different singular points on Γ. Put $R^0(\hat{w}, \bar{k}) = \bigcup_{\bar{w}} R_0(\bar{w}, \bar{k})$ where the first n coordinates of $\bar{w}$ are fixed and coincide with $\hat{w}$, and the last $l - n$ coordinates run over all $(l - n)$-tuples of different regular points on Γ. By construction, $R_0(\bar{w}, \bar{k})$ depends algebraically on $\bar{w}$ and f. Hence $R^0(\hat{w}, \bar{k})$ depends algebraically on f. Since $S_k(\Gamma)$ is a union of a finite number of sets of type $R^0(\hat{w}, \bar{k})$ (with $k = k_1 + \cdots + k_l$) this yields (i).

Let $x(i)$ be the x-coordinate of w_i. Then the equations (4.2) for $w^0 = w_i$ can be viewed as some equations on the $(k_i - 1)$-jet of g at $x(i)$ and they do not impose any restrictions on higher derivatives of g. The existence of Lagrange polynomials implies that when $k \leq m$ the codimension of the $R_0(\bar{w}, \bar{k})$ in G is at least $\sum_i (\log k_i)/\alpha + (l - n)$. Since $k_i \geq 2$ we have $k_1 + \cdots + k_l \leq k_1 \cdots k_l$. Hence the codimension of $R^0(\hat{w}, \bar{k})$ in G is at least $(\log k)/\alpha$, which is (ii).

Let $g^0 \notin S_k(\Gamma)$. Suppose that L_{g^0} meets Γ non-normally at the points from $\bar{w}$ only. By the Lagrange theorem, there exists a nonzero polynomial ω of degree k such that it has zero of order k_i at each point $x(i)$. Furthermore, if we allow the degree of ω to be $k+1$ then we can also suppose that $\omega(0) = 0$. Note that $\omega \in G_m$ by (1). Consider $g := g^0 + \tilde{g}\omega \in G_m$. For generic $\tilde{g}$ the curve L_g meets Γ non-normally at the points of $\bar{w}$ only. By (2) the intersection numbers $L_g \cdot \Gamma$ and $L_{g^0} \cdot \Gamma$ are the same, whence L_g meets Γ at the same number of points as L_{g^0} does, which is (iii). Now (iv) follows from Lemma 4.5. ∎

Corollary 4.9: *Let, under the assumption of Proposition 4.8, the curve $\Gamma = \Gamma_p$ depend algebraically on parameter $p \in P$. Let $M = \max_p \alpha(\Gamma_p)$. Suppose that $m >> \exp(M \dim P)$. Then for generic $g \in G_m$ and every $p \in P$ the defect of L_g relative to Γ_p is at most $m - 2$. In particular, for generic g the curve L_g is Γ_p compatible for every p such that Γ_p does not contain lines parallel to the y-axis.*

5. Main result

We shall consider first the three-dimensional case.

Proposition 5.1: *Let $\{H_p\}$ be a family of hypersurfaces in $\mathbb{C}^3$ with parameter $p \in P$. Then there exists a coordinate system in $\mathbb{C}^3$ such that some plane in this system is strictly compatible relative to $\{H_p\}$.*

Proof: Let (x, y, z) be a coordinate system in $\mathbb{C}^3$ such that none of the surfaces H_p contains a plane $z = \text{const}$ (it is enough to require that the restriction of the projection $(x, y, z) \to (x, z)$ to every H_p is finite, which can be done by Lemma 4.1). We are looking for a new coordinate system of the form

$$(x + \psi(z), y + h(x + \psi(z)), z)$$

where ψ and h are polynomials in one variable whose degree satisfies

$$\deg h >> r = \deg \psi >> \max_p(\deg f_p, \dim P).$$

Consider the variety Q of all pairs $q = (p, \psi)$ where $\psi \in G_r$ (recall that G_r is the set of polynomials of degree at most r) and $p \in P$. Put $f^q(x,y,z) = f_p(x + \psi(z), y, z)$ where f_p is a defining polynomial for H_p. Consider the family of hypersurfaces $\{H^q \mid q \in Q\}$ where H^q is the zero locus of f^q. If Q_0 is a subvariety of Q that consists of $q = (p, \psi_0)$ for some fixed ψ_0 then the subfamily $\{H^q \mid q \in Q_0\}$ coincides with $\{H_p \mid p \in P\}$ after a polynomial coordinate substitution.

Let $\Gamma(q)$ be the image in the (x,y)-plane of the ramification set of the restriction of ρ to the surface given by $f^q(x,y,z) = 0$, and let $\alpha(\Gamma)$ have the same meaning for a curve Γ in the (x,y)-plane as before Proposition 4.8. Put $M = \max_{q\in Q} \alpha(\Gamma(q))$. We require that $\deg h >> \exp(M \dim Q))$.

If we switch from coordinates (x,y,z) to $(x, y + h(x), z)$ the equation of H^q becomes $f^q(x, y+h(x), z) = f_p(x+\psi(z), y+h(x+\psi(z)), z) = 0$. By Lemma 4.1 in this new coordinate system the restriction of the projection $\rho\colon \mathbb{C}^3 \to \mathbb{C}^2$ (given by $(x,y,z) \to (x,y)$) to every H^q is finite. When h is fixed the image of the ramification set of this restriction is a curve Γ^q in the (x,y)-plane which depends on $q \in Q$. Consider polynomials $g \in G_m$ where $deg\, h >> m >> \exp(M \dim Q)$ and a generic polynomial h^0 of the same degree as h. By Lemma 4.6 for every q the intersection number of Γ^q and the curve $y + h^0(x) + g(x) = 0$ is finite and does not depend on g. In particular, this curve is not a component of Γ^q. Replace h by $h + h^0$. Then each curve Γ^q in the (x,y)-plane must be replaced by its image under automorphism $(x,y) \to (x, y - h^0(x))$. Hence after this automorphism none of the curves L_g given by $y + g(x) = 0$ (where $g \in G_m$) is a component of Γ^q. This yields condition (2) from Definition 3.1 for (x,z)-plane A_0 with respect to the family of surfaces $\{H^q | q \in Q\}$.

Note that Γ^q can be obtained from $\Gamma(q)$ by an automorphism of the (x,y)-plane and, therefore, $M = \max_{q\in Q} \alpha(\Gamma^q)$ since α is invariant under automorphisms. By Corollary 4.9 there exists $g^0 \in G_m$ such that for every $q \in Q$ the defect of L_{g^0} relative to Γ^q is at most $m-1$. Without loss of generality suppose that $g^0 = 0$, i.e. $L_0 = L_{g^0}$. By Proposition 4.8 for every $q \in Q$ there exists a nonzero polynomial g such that $g(0) = 0$ and the line L_0 (i.e., the x-axis) meets Γ^q at the same number of points (counting without multiplicity) as the curve L_{cg} for generic $c \in \mathbb{C}$. This gives conditions (3) from Definition 3.1 for A_0 with respect to the family $\{H^q \mid q \in Q\}$.

Consider the family of curves Θ_q in A_0 which are the intersections $H^q \cap A_0$. That is, the equation of Θ_q is $f_p(x + \psi(z), h(x + \psi(z)), z) = 0$. Let k be natural such that $r >> k >> \exp(\dim P \max_p(\deg f_p))$. The intersection number of

Θ_q and a curve $x + \varphi(z) = 0$ with $\varphi \in G_k$ coincides with the degree of the polynomial $f_p(\psi(z) - \varphi(z), h(\psi(z) - \varphi(z)), z)$. That is, the intersection number is independent of φ (though it may depend on p).

If Σ is a curve in $\mathbb{C}^2$ denote by $\mu(\Sigma)$ the maximum of multiplicities of its points. Consider the curve Σ in A_0 given by the equation $f_p(x, h(x), z) = 0$. Note that $\mu(\Sigma)$ is at most $\deg_z f_p(x, y, z) \leq \deg f_p$ since this is the number of points at which a generic line $x =$ const meets Σ. Since Θ_q can be obtained from Σ by an automorphism we see that $\max_{q \in Q} \mu(\Theta_q)$ is bounded by $\max_p \deg f_p$. Hence $\max_{q \in Q} \alpha(\Theta_q)$ is bounded by $4 \log(\max_p \deg f_p)$.

By Proposition 4.8 for every fixed q there exists a subset $S \subset G_k$ of codimension $>> \dim P$ such that for every $\varphi \in G_k \setminus S$ the defect of Θ_q relative to the curve $x + \varphi(z) = 0$ is at most $k - 2$. This is equivalent to the fact that the sum of the multiplicities of the multiple roots of the polynomial

$$f_p(\psi(z) - \varphi(z), h(\psi(z) - \varphi(z)), z)$$

is at most $k-1$. Replacing $\psi(z)$ by $\psi(z) - \varphi(z)$ one can see that for generic ψ the sum of the multiplicities of the multiple roots of the polynomial $f_p(\psi(z), h(\psi(z)), z)$ is at most $k - 2$. More precisely, this is true for every ψ except for a subvariety of G_r of codimension $>> \dim P$ (and this subvariety depends, of course, algebraically on p). This implies that there exists $\psi_1 \in G_r$ such that for every $p \in P$ the sum of the multiplicities of the multiple roots of the polynomial $f_p(\psi_1(z), h(\psi_1(z)), z)$ is at most $k - 2$. Note that for every p this sum is defect of Θ_q with respect to the curve $x + \varphi(z) = 0$ where $q = (p, \psi)$ and $\psi = \psi_1 + \varphi$.

Fix φ_0 and let $\psi_0 = \psi_1 + \varphi_0$. Consider the subvariety Q_0 of Q as in the beginning of the proof. Since the intersection number of Θ_q and any curve $x + \varphi(z) = 0$ (with $\varphi \in G_k$) is independent of φ, in order to show that the curve $x + \varphi_0(z) = 0$ is Θ_q compatible for every $q \in Q_0$ it suffices to show that none of the curves Θ_q contains a line $z = c$ (see Proposition 4.8). Consider the curves $\{\Lambda_{c,q}\}$ in the (x, y)-plane given by the equations $f_p(x + \psi(c), y + h(x + \psi(c)), c) = 0$. Note that Θ_q contains the line $z = c$ iff $\Lambda_{c,q}$ contains the x-axis. But $\Lambda_{c,q}$ cannot contain the x-axis since the restriction of the projection $(x, y) \to y$ to every $\Lambda_{c,q}$ is finite by Lemma 4.1.

Hence $x + \varphi_0(z) = 0$ is Θ_q compatible for every $q \in Q_0$. That is, condition (1) from Definition 3.1 is true for A_0 with respect to the subfamily $\{H^q \mid q \in Q_0\}$. Thus A_0 is strictly compatible relative to the family $\{H^q \mid q \in Q_0\}$ by Theorem 3.2. ∎

Remark 5.2: It is not difficult to check that every plane from a Zariski open neighborhood of A_0 in the variety of planes (i.e., every generic plane) is strictly compatible with respect to the subfamily $\{H^q \mid q \in Q_0\}$. In fact, one can construct more sophisticated coordinate substitutions such that every plane becomes strictly H_p compatible for every $p \in P$. The same remark is applicable in the case of an arbitrary dimension.

LEMMA 5.3: *Let H be a hypersurface in $\mathbb{C}^{n+1}$, let λ be a coordinate function on $\mathbb{C}^{n+1}$, and let R be a hyperplane in $\mathbb{C}^{n+1}$ which is not given by $\lambda = \text{const}$. For $c \in \mathbb{C}$ put $C_c = \lambda^{-1}(c)$, $R_c = C_c \cap R$, $H_c = H \cap C_c$. Suppose that for a generic $c \in \mathbb{C}$ the manifold R_c is strictly H_c compatible in $C_c \cong \mathbb{C}^n$ and that for every $c \in \mathbb{C}$ the set R_c is not an irreducible component of H_c. Then R is strictly H compatible.*

Proof: Let S be a finite set in $\mathbb{C}$. Put $R(S) = R - (\lambda^{-1}(S) \cup H)$, $C(S) = \mathbb{C}^{n+1} - (\lambda^{-1}(S) \cup H)$, $\lambda_S = \lambda|_{C(S)}$, and $\lambda'_S = \lambda|_{R(S)}$. Choose the finite set $S \subset \mathbb{C}$ so that the mappings λ_S: $C(S) \to \mathbb{C} - S$ and λ'_S: $R(S) \to \mathbb{C} - S$ are fibrations. Let F be the fiber of λ_S and let F' be the fiber λ'_S. Then, by assumption, the natural embedding e: $F' \to F$ generates an isomorphism e_*: $\pi_1(F') \to \pi_1(F)$. We have also two other embeddings i: $F \to C(S)$ and i': $F' \to R(S)$ which implies the commutative diagram

$$\begin{array}{ccccccc} 0 \to & \pi_1(F) & \overset{i_*}{\to} \pi_1(C(S)) & \overset{\lambda_{S*}}{\to} & \pi_1(\mathbb{C}-S) & \to 0 \\ \uparrow & \uparrow e_* & \uparrow & & \uparrow id & \uparrow \\ 0 \to & \pi_1(F') & \overset{i'_*}{\to} \pi_1(R(S)) & \overset{\lambda'_{S*}}{\to} & \pi_1(\mathbb{C}-S) & \to 0. \end{array}$$

The five isomorphisms lemma implies that $\pi_1(C(S))$ and $\pi_1(R(S))$ are isomorphic. Since R_s is not contained in H_s for every s one can choose simple loops γ_s in $R(S)$ around each hypersurface R_s with $s \in S$ such that γ_s is contractible in $R - H$. Consider the natural embedding j': $R(S) \to R - H$. It generates an epimorphism j'_*: $\pi_1(R(S)) \to \pi_1(R - H)$ and, obviously, $[\gamma_s] \in \ker j'_*$ for every $s \in S$. Moreover, if N' is the smallest normal subgroup in $\pi_1(R(S))$ that contains all $[\gamma_s]$, $s \in S$, then $\ker j'_* = N'$, which follows from two simple geometrical observations:

—each two-cell in $R - H$ becomes transversal to $\lambda^{-1}(S) \cap R$ after a perturbation, i.e. every contractible loop in $R - H$ can be viewed as a product of simple contractible loops around hyperplanes R_s, $s \in S$;

—every simple contractible loop of this type is conjugate to some $[\gamma_s]$ as an element of $\pi_1(R(S))$.

Similarly, we can consider the embedding $j: C(S) \to \mathbb{C}^n - H$. It generates an epimorphism $j_*: \pi_1(C(S)) \to \pi_1(\mathbb{C}^n - H)$. Then $\ker j_*$ coincides with the smallest normal subgroup N of $\pi_1(C(S))$ that contains all $[\gamma_s]$, $s \in S$, where we treat $[\gamma_s]$ as an element of $\pi_1(C(S))$ now. This yields an isomorphism between $\pi_1(R - H)$ and $\pi_1(\mathbb{C}^n - H)$, which concludes the proof. ∎

THEOREM 5.4: *Let $\{H_p\}$ be a family of hypersurfaces in $\mathbb{C}^n$ with parameter $p \in P$. Then there exists a polynomial coordinate system in $\mathbb{C}^{n+1}$ such that some plane is strictly H_p compatible for every $p \in P$.*

Proof: We shall use induction. The first step of induction is Proposition 5.1. Assume that for every family of hypersurfaces in $\mathbb{C}^n$ there exists a coordinate system such that some hyperplane in this system is strictly compatible relative to this family. Consider a family of hypersurfaces $\{H_p\}$ in $\mathbb{C}^{n+1}$. Let $\bar{x} = (x_1, \ldots, x_{n+1})$ be a coordinate system in $\mathbb{C}^{n+1}$ and let λ coincide with x_1 on $\mathbb{C}^{n+1}$. We can choose $\bar{x}$ so that none of H_p contains a hyperplane $x_1 = \text{const}$ since the restriction of the projection $\bar{x} \to (x_1, \ldots, x_n)$ to H_p can be supposed to be finite, by the analogue of Lemma 4.1 in the case of higher dimensions. Put $C_c = \lambda^{-1}(c)$. We can view $H_{p,c} = H_p \cap C_c$ as a hypersurface in the fiber C_c. Put $Q = P \times \mathbb{C}$. Then we can consider $\{H_{p,c} = H^q\}$ with $q = (p, c) \in Q$ as a family of hypersurfaces in $\mathbb{C}^n$. By induction, a coordinate system $\bar{y} = (y_2, \ldots, y_{n+1})$ in $\mathbb{C}^n$ can be chosen so that some hyperplane E in $\mathbb{C}^n$ is strictly compatible relative to $\{H^q\}$. In particular, none of the hypersurfaces H^q contains E. Let R be the hyperplane $\tau^{-1}(E)$ in the coordinate system $(x_1, \bar{y})$ in $\mathbb{C}^{n+1}$ where τ is the natural projection $(x_1, \bar{y}) \to \bar{y}$. By Lemma 5.3, the hyperplane R is strictly compatible relative to $\{H_p\}$. Therefore, we can reduce dimension by induction, which implies our Theorem. ∎

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